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Global Nonexistence of Smooth Electric Induction Fields
in Nonlinear Dielectrics¹

I. Infinite Cylindrical Dielectrics

by

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conductor, it is shown that, under relatively mild conditions on λ , solutions of the corresponding initial-boundary value problem for the electric induction field cannot exist globally in time in L_2 sense if it is assumed that the electric field in the rod is perpendicular to the axis of the rod and varies as the coordinate along that axis. It is also shown that, when the initial electromagnetic field in the rod has a compact support, Riemann Invariant arguments may be applied to show that the space-time gradient of the non-zero component of the electric induction field must blow-up in finite time. Some growth estimates for solutions, which are valid on the maximal time-interval of existence are also derived; these are valid in the simple but physically important case where $\lambda(D) = \lambda_1 + \lambda_0 \|D\|^2$. We also discuss relations with recent work on the phenomena of self-focusing and self-trapping for high intensity laser beams in a dielectric medium.

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Global Nonexistence of Smooth Electric Induction Fields
in Nonlinear Dielectrics¹

ABSTRACT

Coupled nonlinear wave equations are derived for the evolution of the components of the electric induction field \underline{D} in a class of rigid nonlinear dielectrics governed by the nonlinear constitutive relation $\underline{E} = \lambda(\underline{D})\underline{D}$, where \underline{E} is the electric field and $\lambda > 0$ is a scalar-valued vector function. For the special case of an infinite one-dimensional dielectric rod, embedded in a perfect conductor, it is shown that, under relatively mild conditions on λ , solutions of the corresponding initial-boundary value problem for the electric induction field can not exist globally in time in the L_2 sense if it is assumed that the electric field in the rod is perpendicular to the axis of the rod and varies as the coordinate along that axis. It is also shown that, when the initial electromagnetic field in the rod has compact support, Riemann Invariant arguments may be applied to show that the space-time gradient of the non-zero component of the electric induction field must blow-up in finite time. Some growth estimates for solutions, which are valid on the maximal time-interval of existence are also derived; these are valid in the simple but physically important case where $\lambda(D) = \lambda_1 + \lambda_0 \|D\|^2$. We also discuss relations with recent work on the phenomena of self-focusing and self-trapping for high intensity laser beams in a dielectric medium.

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MATTHEW J. KERFER
Chief, Technical Information Division

1. Evolution Equations for a Class of Nonlinear Dielectrics

Theories of material dielectric behavior are based upon a set of field equations (Maxwell's equations) and a set of constitutive relations which hold among the electromagnetic field vectors. In a Lorentz reference frame (x^i, t) , $i = 1, 2, 3$, where the (x^i) represent rectangular Cartesian coordinates, and t is the time parameter, the local forms of Maxwell's equations are given by

$$(1.1) \quad \begin{cases} -\frac{\partial \underline{B}}{\partial t} + \text{curl } \underline{E} = \underline{0}, & \text{div } \underline{E} = 0, \\ \text{curl } \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{0}, & \text{div } \underline{D} = 0, \end{cases}$$

provided that the density of free current, the magnetization, and the density of free charge all vanish. In (1.1), \underline{B} , \underline{E} , and \underline{H} are, respectively, the magnetic flux density, electric field, and magnetic intensity while $\underline{D} \equiv \epsilon_0 \underline{E} + \underline{P}(\underline{E})$, $\epsilon_0 > 0$ a physical constant and \underline{P} the polarization vector, is the electric induction field; the relations (1.1) hold in some bounded open domain $\Omega \subseteq \mathbb{R}^3$ which is filled with a rigid, nonconducting, dielectric substance. The precise nature of the dielectric medium in Ω is determined by specifying a set of constitutive equations relating \underline{E} , \underline{D} , \underline{H} , and \underline{B} ; indeed, without the specification of additional relations among the electromagnetic field vectors, the set of equations (1.1) represents an indeterminate system.

There is, in existence, a wide variety of constitutive hypotheses which have been associated with theories of nonconducting, rigid, dielectric media; the simplest of these is that associated with the dielectric response of a vacuum in which there hold the classical constitutive relations

$$\underline{D} = \epsilon_0 \underline{E}, \quad \underline{H} = \mu_0^{-1} \underline{B},$$

where the fundamental physical constants ϵ_0, μ_0 satisfy $\epsilon_0 \mu_0 = c^{-2}$, c being the speed of light in a vacuum. In 1873 Maxwell [1] proposed as a set of constitutive laws for a linear, rigid, stationary non-conducting dielectric the relations

$$\underline{D} = \underline{\epsilon} \cdot \underline{E}, \quad \underline{B} = \underline{\mu} \cdot \underline{H}$$

where $\underline{\epsilon}, \underline{\mu}$ are constant second-order tensors which are proportional to the identity tensor if the material is isotropic. A set of constitutive relations, which are still linear, but which take into account certain memory effects in the dielectric, were proposed by Maxwell in 1877 and subsequently used by Hopkinson [2] in connection with his studies on the residual charge of the Leyden jar; the Maxwell-Hopkinson dielectric is governed by the set of constitutive relations ($\underline{x} \in \Omega$):

$$(1.2) \quad \begin{cases} \underline{D}(\underline{x}, t) = \epsilon \underline{E}(\underline{x}, t) + \int_{-\infty}^t \phi(t-\tau) \underline{E}(\underline{x}, \tau) d\tau \\ \underline{H} = \mu^{-1} \underline{B} \end{cases}$$

where $\epsilon > 0, \mu > 0$ and $\phi(t), t \geq 0$ is a continuous monotonically decreasing function of $t, 0 \leq t < \infty$. Noting that the Maxwell-Hopkinson constitutive relations do not account for the observed absorption and dispersion of electromagnetic waves in material non-conductors, Toupin and Rivlin [3] generalized the constitutive relations (1.2) and introduced the concepts of holohedral isotropic and hemihedral dielectric response; while the response incorporated into both of these theories is linear, they are more sophisticated than (1.2) in the sense that magnetic memory effects and coupling of electric and magnetic effects is built into the constitutive theory. The qualitative behavior of the electric induction field in a rigid non-conducting dielectric exhibiting holohedral isotropic response has been studied by this author in a series of recent papers [4] - [8].

In this paper we will be concerned with initial-boundary value problems associated with the evolution of the components of the electric induction field \underline{D} in a relatively simple class of materials exhibiting nonlinear dielectric response. A rather general theory of nonlinear dielectric behavior which allows for both electric and magnetic memory effects, but still effects an a priori separation of electric and magnetic response, was proposed by Volterra [7] in 1912 in the form of the constitutive relations

$$(1.3) \quad \begin{cases} \underline{D}(\underline{x}, t) = \underline{\epsilon} \cdot \underline{E}(\underline{x}, t) + \int_{-\infty}^t \underline{D}(\underline{E}(\underline{x}, t)), \quad \underline{x} \in \Omega \\ \underline{B}(\underline{x}, t) = \underline{\mu} \cdot \underline{H}(\underline{x}, t) + \int_{-\infty}^t \underline{B}(\underline{H}(\underline{x}, t)), \quad \underline{x} \in \Omega \end{cases}$$

The constitutive relations (1.3) reduce to those considered in [2], [3] under special assumptions relative to the functionals \underline{D} , \underline{B} , i.e., if $\underline{B} = 0$, \underline{D} is linear and isotropic, and $\underline{\epsilon} = \epsilon \underline{I}$, $\underline{\mu} = \mu \underline{I}$, then (1.3) is easily seen to reduce to (1.2); the particular class of nonlinear dielectrics to be considered in this exposition results by specializing (1.3) to the situation where $\underline{\mu} = \mu \underline{I}$, $\mu > 0$, $\underline{B} = 0$, and electric field memory effects are negligible, i.e.,

$$(1.4a) \quad \begin{cases} \underline{D}(\underline{x}, t) = \underline{D}(\underline{E}(\underline{x}, t)), \quad \underline{x} \in \Omega \\ \underline{B}(\underline{x}, t) = \mu \underline{H}(\underline{x}, t), \quad \underline{x} \in \Omega \end{cases}$$

We shall further assume that $\det \left[\frac{\partial D_i}{\partial E_j} \right]_{\underline{E}=\underline{0}} \neq 0$, so that in a (Euclidean) neighborhood of $\underline{E} = \underline{0}$, the relations (1.4a) may be inverted so as to yield the constitutive equations

$$(1.4b) \quad \begin{cases} \underline{E}(\underline{x}, t) = \underline{E}(\underline{D}(\underline{x}, t)), \quad \underline{x} \in \Omega \\ \underline{H}(\underline{x}, t) = \mu^{-1} \underline{B}(\underline{x}, t), \quad \underline{x} \in \Omega \end{cases}$$

As the vector function \underline{E} is still completely arbitrary, the constitutive theory defined by (1.4b) is still far too general to provide a tractable system of evolution equations for the electromagnetic field in Ω ; we will, therefore, confine our attention to that special case of (1.4b) for which there exists a scalar-valued vector function $\underline{\lambda}(\underline{E})$ such that $\underline{E}(\underline{E}) = \underline{\lambda}(\underline{E})\underline{E}$, $\forall \underline{E}$ with real components E_i . Thus, the final form of the constitutive relations which define the nonlinear dielectric response to be considered here is given by

$$(1.5) \quad \begin{cases} \underline{D}(\underline{x}, t) = \underline{\lambda}(\underline{D}(\underline{x}, t))\underline{D}(\underline{x}, t), & \underline{x} \in \Omega \\ \underline{H}(\underline{x}, t) = \mu^{-1} \underline{B}(\underline{x}, t), & \underline{x} \in \Omega \ (\mu > 0) \end{cases}$$

For now we will simply assume that $0 \leq \underline{\lambda}(\underline{E}) < \infty$, $\forall \underline{E}$, with $\underline{\lambda}(\underline{E}) > 0$, $\forall \underline{E} \neq 0$; further assumptions on the constitutive function $\underline{\lambda}$ will be imposed below.

Remarks It seems worthwhile to note, in passing, that electromagnetic constitutive relations of the form (1.5) or, to be somewhat more accurate, the inverted relations

$$(1.6) \quad \begin{cases} \underline{D}(\underline{x}, t) = \underline{\epsilon}(\underline{E}(\underline{x}, t))\underline{E}(\underline{x}, t), & \underline{x} \in \Omega \\ \underline{B}(\underline{x}, t) = \mu \underline{H}(\underline{x}, t), & \underline{x} \in \Omega \ (\mu > 0) \end{cases}$$

have appeared in the recent literature; e.g., Rivlin [8] considers (1.6) and indicates that in an isotropic material conforming to this constitutive hypothesis the dielectric "constant" $\underline{\epsilon}$ must be an even function of the magnitude of \underline{E} , i.e., $\underline{\epsilon} = \underline{\epsilon}(\underline{E} \cdot \underline{E})$. Townes, et. al. [14] considered the problem of a high intensity Laser beam propagating through a dielectric medium; they assume that the high intensity of the beam affects the dielectric "constant" $\underline{\epsilon}$ in such a way that the effective $\underline{\epsilon}$ in the medium is given by $\underline{\epsilon} = \underline{\epsilon}_0 + \underline{\epsilon}_2 \|\underline{E}\|^2$ where $\underline{\epsilon}_0 > 0$, $\underline{\epsilon}_2 > 0$; they then go on to demonstrate that the presence of the nonlinearity

may give rise to an electromagnetic beam which produces its own wave guide and thus propagates without spreading (the so-called phenomena of self-trapping of the beam). Strauss [16] and Whitham [17] both consider a polarized wave with frequency ω propagating in the direction \underline{e} (parallel to the x_3 -axis in our cartesian coordinate system) in a dielectric. They assume that the high intensity of the electromagnetic field in the beam, given by

$$\underline{E} = u(x_1, x_2, x_3) e^{ikx_3} e^{-i\omega t} \underline{e}$$

again produces changes in the dielectric constant ϵ so that $\epsilon = \epsilon_0 + \epsilon_2 \|\underline{E}\|^2$. Using a paraxial approximation, i.e., $|u_{x_3 x_3}| \ll |k u_{x_3}|$, $k = \omega \sqrt{\epsilon_0} / c$, these authors ([16], [17]) then claim that $u(x_1, x_2, x_3)$ satisfies a nonlinear Schrödinger equation of the form

$$2ik \frac{\partial u}{\partial x_3} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - k \frac{\epsilon_2}{\epsilon_0} |u|^2 u = 0$$

Using this last equation (and setting $x_3 = t$) these authors show that under an appropriate set of assumptions the intensity $|u|^2$ of the beam blows up at a finite value of t and thus claims to have a rigorous demonstration of the phenomena of self-focusing of an electromagnetic beam. We will indicate, following the statement and proof of our first Lemma below, why we feel that the reductions of the pertinent evolution equations for the electromagnetic field in the beam, to the nonlinear Schrödinger equation (given above) for the intensity $u(x_1, x_2, x_3)$, are in error and ignore, in effect, the basic nonlinear character of the dielectric medium in which the beam is propagating.

Lemma 1. Let $\Omega \subseteq \mathbb{R}^3$ be either a bounded or unbounded domain and assume that Ω is filled with a rigid, nonlinear, nonconducting dielectric substance which conforms to the constitutive hypothesis (1.5). Then, in Ω , the components $D_i(x, t)$, of the electric induction field, satisfy the coupled system of nonlinear wave equations

$$(1.7) \quad \mu \frac{\partial^2 D_i}{\partial t^2} = \nabla^2 (\lambda(D) D_i) - \frac{\partial}{\partial x_i} (\text{grad } \lambda(D) \cdot D), \quad i = 1, 2, 3.$$

Proof. We begin with the identity

$$\Delta A = \text{grad}(\text{div } A) - \text{curl } \text{curl } A$$

which is valid for any sufficiently smooth vector field on Ω ; applied to the electric field $E(\cdot, t)$ the identity yields

$$(1.8) \quad \nabla^2 E_i = \frac{\partial}{\partial x_i} (\text{div } E) - (\text{curl } \text{curl } E)_i; \quad i = 1, 2, 3.$$

In view of Maxwell's equations (1.1), and the second constitutive relation in (1.5), we have

$$\begin{aligned} \text{curl } \text{curl } E &= - \text{curl } \left(\frac{\partial B}{\partial t} \right) \\ &= - \mu \text{curl } \left(\frac{\partial H}{\partial t} \right) \\ &= - \mu \frac{\partial}{\partial t} (\text{curl } H) \\ &= - \mu \frac{\partial^2 D}{\partial t^2}, \end{aligned}$$

so that (1.8) has the equivalent form

$$(1.9) \quad \mu \frac{\partial^2 D_i}{\partial t^2} = \nabla^2 E_i - \frac{\partial}{\partial x_i} (\text{div } E), \quad i = 1, 2, 3$$

By (1.4b),

$$\begin{cases} \text{div } E = \frac{\partial F_j}{\partial x_j} \cdot \frac{\partial D_k}{\partial x_j} = A_{jk}(D) \frac{\partial D_k}{\partial x_j} \\ \nabla^2 E_i = \frac{\partial}{\partial x_k} \left(\frac{\partial F_i}{\partial x_k} \right) = \frac{\partial}{\partial x_k} (A_{ik}(D) \frac{\partial D_k}{\partial x_k}), \end{cases}$$

where $A_{ij}(\underline{D}) \equiv \frac{\partial E_i}{\partial D_j}$, and the standard summation convention has been employed. Thus (1.9) becomes

$$(1.10) \quad \mu \frac{\partial^2 D_i}{\partial t^2} = \frac{\partial}{\partial x_k} (A_{ik}(\underline{D}) \frac{\partial D_k}{\partial x_i}) - \frac{\partial}{\partial x_i} (A_{ijk}(\underline{D}) \frac{\partial D_k}{\partial x_j})$$

However, by virtue of our hypothesis that $E_i(\underline{D}) = \lambda(\underline{D}) D_i$, we easily find that

$$A_{ij}(\underline{D}) = \lambda(\underline{D}) \delta_{ij} + \frac{\partial \lambda}{\partial D_j} D_i$$

and therefore

$$(1.11) \quad \begin{aligned} \mu \frac{\partial^2 D_i}{\partial t^2} &= \frac{\partial}{\partial x_k} \left(\left[\lambda(\underline{D}) \delta_{ik} + \frac{\partial \lambda}{\partial D_k} D_i \right] \frac{\partial D_k}{\partial x_i} \right) \\ &\quad - \frac{\partial}{\partial x_i} \left(\left[\lambda(\underline{D}) \delta_{jk} + \frac{\partial \lambda}{\partial D_k} D_j \right] \frac{\partial D_k}{\partial x_j} \right) \end{aligned}$$

where we sum on each repeated index; expanding (1.11) and using the Maxwell relation $\text{div } \underline{D} = \frac{\partial D_i}{\partial x_i} = 0$, we obtain the stated result (1.7), i.e.,

$$(1.12) \quad \begin{aligned} \mu \frac{\partial^2 D_i}{\partial t^2} &= \frac{\partial}{\partial x_k} \left(\lambda(\underline{D}) \frac{\partial D_i}{\partial x_k} \right) + \frac{\partial}{\partial x_k} \left(\frac{\partial \lambda}{\partial x_k} D_i \right) - \frac{\partial}{\partial x_i} \left(\frac{\partial \lambda}{\partial x_j} D_j \right) \\ &= \frac{\partial^2}{\partial x_k \partial x_k} (\lambda(\underline{D}) D_i) - \frac{\partial}{\partial x_i} \left(\frac{\partial \lambda}{\partial x_j} D_j \right) \end{aligned}$$

Remarks

We now return to the discussion of the work of Strauss [16], and Whitham [15] which we began prior to the statement of the Lemma above.

If we take the constitutive equations (1.5) in the inverted form (1.6) and substitute into (1.9) we obviously obtain

$$\mu \frac{\partial^2}{\partial t^2} (\epsilon(\underline{E}) E_i) = \nabla^2 E_i - \partial / \partial x_i (\text{div } \underline{E})$$

However, in a nonlinear dielectric media it is not generally true that $\text{div } \underline{E} = 0$ and thus the term $\partial / \partial x_i (\text{div } \underline{E})$ can not be discarded in the above evolution

equation for the electric field. In particular, if $\underline{D} = \epsilon(\underline{E})\underline{E} = (\epsilon_0 + \epsilon_2 \|\underline{E}\|^2)\underline{E}$ then $\text{div} [(\epsilon_0 + \epsilon_2 \|\underline{E}\|^2)\underline{E}] = 0$ and not $\text{div } \underline{E} = 0$. In [15] however (page 549) the author has tacitly assumed that $\text{div } \underline{E} = 0$ even though he proceeds to employ a nonlinear constitutive relation between \underline{P} and \underline{E} (and, thus, between \underline{D} and \underline{E}) while in [16] the author begins with the standard Electromagnetic wave equation,

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\epsilon \underline{E}) = \nabla^2 \underline{E}$$

assumes the form $\underline{E} = u(x_1, x_2, x_3) e^{ikx_3} e^{-i\omega t} \underline{e}$ for the wave, so as to reduce the last equation to an equation for $u(x_1, x_2, x_3)$ of the form

$$\partial_{ik} \frac{\partial u}{\partial x_3} - \Delta u + (k^2 - \frac{\epsilon \omega^2}{c^2}) u = 0,$$

and then assumes that the high intensity of the (laser) beam modifies the dielectric character of the beam so that $\epsilon = \epsilon_0 + \epsilon_2 \|\underline{E}\|^2$; this form for ϵ is then substituted into the last equation for $u(x_1, x_2, x_3)$ so as to give (modulo an approximation) the "appropriate" nonlinear Schrodinger equation for u . The problem with all of this is that at that point at which the beam has modified the character of the dielectric medium in which it is propagating, so that

$$\epsilon = \epsilon_0 + \epsilon_2 \|\underline{E}\|^2$$

it is no longer true that $\text{div } \underline{E} = 0$ and the standard equation $\frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\epsilon \underline{E}) = \nabla^2 \underline{E}$ is no longer valid, i.e., the assumed form for \underline{E} , $\underline{E} = u e^{ikx_3} e^{-i\omega t} \underline{e}$ must be substituted into the more general equation

$$\mu \frac{\partial^2}{\partial t^2} (\epsilon(\underline{E}) \underline{E}) = \nabla^2 \underline{E} - \partial / \partial x_1 (\text{div } \underline{E})$$

with $\epsilon(E) = \epsilon_0 + \epsilon_2 |E|^2$. Under these conditions the Schrödinger equation derived in [15], [16] for the intensity $u(x_1, x_2, x_3)$ will clearly not result and does not seem to follow from any reasonable set of approximations; a rigorous demonstration of self-focusing for the beam described by $\underline{E} = u e^{ikx_1} e^{-i\omega t} \underline{e}$ would therefore seem to be an open problem.

We now assume that $\partial\Omega$ is sufficiently smooth to admit of applications of the divergence theorem and we denote by $\underline{\nu}(x)$ the exterior unit normal to $\partial\Omega$ at a point $x \in \partial\Omega$; we also denote by $\underline{t}(x)$ a generic vector in the tangent plane to $\partial\Omega$ at $x \in \partial\Omega$. The evolution equations (1.7) are to hold in some cylinder $\Omega \times [0, T)$, $T > 0$, in R^4 and we now associate with this system a set of initial and boundary data. In Ω we require that

$$(1.13) \quad D_i(x, 0) = f_i(x), \quad \frac{\partial D_i}{\partial t}(x, 0) = g_i(x), \quad \begin{cases} x \in \Omega \\ i=1, 2, 3 \end{cases}$$

Standard results from electromagnetic theory [9, §13] also dictate that if Ω is a bounded domain in R^3 then

$$(1.14a) \quad [D(x, t) \cdot \underline{\nu}(x)] = \sigma(x), \quad (x, t) \in \partial\Omega \times [0, T)$$

$$(1.14b) \quad [\underline{D}(x, t) \cdot \underline{t}(x)] = 0, \quad (x, t) \in \partial\Omega \times [0, T)$$

In the set of relations (1.14), $[F(x)]$ denotes the jump of the scalar-valued function F across $\partial\Omega$ at $x \in \partial\Omega$ while $\sigma(x)$ denotes the density of surface charge at the point $x \in \partial\Omega$; these boundary conditions can be written in an alternative form as follows: If we let $\underline{D}^*(x, t)$ denote the electric induction field at points $(x, t) \in R^3/\Omega \times [0, T)$ then (1.14a), (1.14b) are clearly equivalent to

$$(1.15a) \quad \underline{D}(\underline{x},t) \cdot \underline{\nu}(\underline{x}) - \underline{D}^*(\underline{x},t) \cdot \underline{\nu}(\underline{x}) = \sigma(\underline{x}), (\underline{x},t) \in \partial\Omega \times [0,T)$$

$$(1.15b) \quad \lambda(\underline{D}(\underline{x},t)) \underline{D}(\underline{x},t) \cdot \underline{\tau}(\underline{x}) = \underline{E}^*(\underline{x},t) \cdot \underline{\tau}(\underline{x}), (\underline{x},t) \in \partial\Omega \times [0,T)$$

where $\underline{E}^*(\underline{x},t)$, $(\underline{x},t) \in \mathbb{R}^2/\Omega \times [0,T)$, is the electric field associated with $\underline{D}^*(\underline{x},t)$. In particular, let $\Omega = \tilde{\Omega} \subseteq \mathbb{R}^3$, and $\tilde{\Omega}/\Omega$ is filled with a perfect conductor (in which $\underline{D}^* = \underline{E}^* = \underline{0}$) then (1.15a), (1.15b) reduce to

$$(1.16a) \quad \underline{D}(\underline{x},t) \cdot \underline{\nu}(\underline{x}) = \sigma(\underline{x}), (\underline{x},t) \in \partial\Omega \times [0,T)$$

$$(1.16b) \quad \lambda(\underline{D}(\underline{x},t)) \underline{D}(\underline{x},t) \cdot \underline{\tau}(\underline{x}) = 0, (\underline{x},t) \in \partial\Omega \times [0,T).$$

In this paper we wish to consider that particular subcase of the general initial-boundary value problem (1.7), (1.15), (1.16a,b) which corresponds to the assumption that the geometry of Ω is an infinite one-dimensional (non-linear dielectric rod); we want to investigate whether a smooth electric field, which is perpendicular to the axis of the rod, and depends only on variations of the coordinate along that axis, can exist globally, i.e., for $t \in [0, \infty)$. We assume, therefore, that the rod occupies the configuration depicted in Figure 1, below. The problem of considering a finite rod gives rise, as a consequence of the appropriate specializations of (1.16), to the imposition of a priori smoothness assumptions on $\underline{D}(\underline{x},t)$ at the planar boundaries of the rod. We comment on this situation at the end of §3.

$$(1.15a) \quad \underline{D}(\underline{x}, t) \cdot \underline{v}(\underline{x}) - \underline{D}^*(\underline{x}, t) \cdot \underline{v}(\underline{x}) = \sigma(\underline{x}), \quad (\underline{x}, t) \in \partial\Omega \times [0, T)$$

$$(1.15b) \quad \lambda(\underline{D}(\underline{x}, t)) \underline{D}(\underline{x}, t) \cdot \underline{t}(\underline{x}) = \underline{E}^*(\underline{x}, t) \cdot \underline{t}(\underline{x}), \quad (\underline{x}, t) \in \partial\Omega \times [0, T)$$

where $\underline{E}^*(\underline{x}, t)$, $(\underline{x}, t) \in \mathbb{R}^3/\Omega \times [0, T)$, is the electric field associated with $\underline{D}^*(\underline{x}, t)$. In particular, if $\Omega \subset \tilde{\Omega} \subset \mathbb{R}^3$, and $\tilde{\Omega}/\Omega$ is filled with a perfect conductor (in which $\underline{D}^* = \underline{E}^* = \underline{0}$) then (1.15a), (1.15b) reduce to

$$(1.16a) \quad \underline{D}(\underline{x}, t) \cdot \underline{v}(\underline{x}) = \sigma(\underline{x}), \quad (\underline{x}, t) \in \partial\Omega \times [0, T)$$

$$(1.16b) \quad \lambda(\underline{D}(\underline{x}, t)) \underline{D}(\underline{x}, t) \cdot \underline{t}(\underline{x}) = 0, \quad (\underline{x}, t) \in \partial\Omega \times [0, T).$$

In this paper we wish to consider that particular subcase of the general initial-boundary value problem (1.7), (1.13), (1.16a,b) which corresponds to the assumption that the geometry of Ω is an infinite one-dimensional (non-linear dielectric rod); we want to investigate whether a smooth electric field, which is perpendicular to the axis of the rod, and depends only on variations of the coordinate along that axis, can exist globally, i.e., for $t \in [0, \infty)$. We assume, therefore, that the rod occupies the configuration depicted in Figure 1, below. The problem of considering a finite rod gives rise, as a consequence of the appropriate specializations of (1.16), to the imposition of a priori smoothness assumptions on $\underline{D}(\underline{x}, t)$ at the planar boundaries of the rod. We comment on this situation at the end of §3.

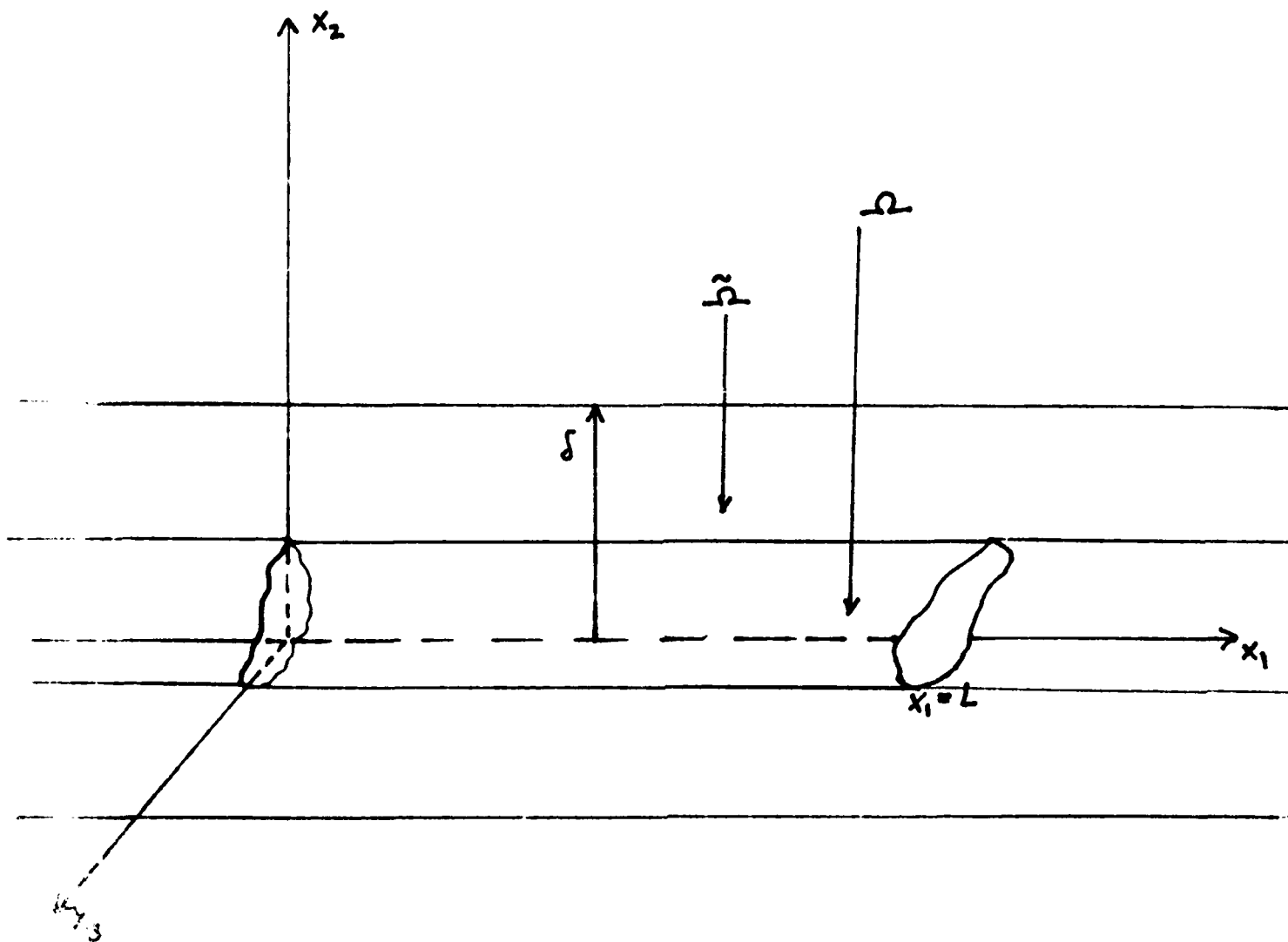


Figure 1

Specifically, we take for Ω the finite cylinder

$$(1.17) \quad \Omega = \{(x_1, x_2, x_3) \mid x_i \text{ real, } i = 1, 2, 3, -\infty < x_1 < \infty,$$

$$\delta(x_2, x_3) = C_1(\text{const.})\}$$

with generators parallel to the x_1 axis and we assume that for some small $\epsilon > 0$

$$\begin{aligned} \Omega \cap \{(x_1, x_2, x_3) \mid -\infty < x_1 < \infty\} \\ \subseteq \{(x_1, x_2, x_3) \mid -\infty < x_1 < \infty, x_2^2 + x_3^2 \leq \epsilon^2\}. \end{aligned}$$

For $\tilde{\Omega}$ we then take the (infinite) circular cylinder

$$(1.18) \quad \tilde{\Omega} = \{(x_1, x_2, x_3) \mid -\infty < x_1 < \infty, x_2^2 + x_3^2 = \delta^2, \delta > \epsilon > 0\}$$

and assume that the annular region $\tilde{\Omega}/\Omega$ between the dielectric rod and the circular cylinder is filled with a perfect conductor; in Ω the dielectric media is assumed to be governed by the constitutive hypothesis (1.5). We want to examine the possibility of there existing in the rod a smooth electric field which is perpendicular to the $x_1 x_3$ plane and hence, orthogonal to the axis of the dielectric; specifically, we are interested in smooth electric fields of the form

$$(1.19) \quad E(x, t) = (0, E_2(x_1, t), 0), \quad -\infty < x_1 < \infty.$$

Of course, in $\tilde{\Omega}/\Omega$ we must have $E = 0$. In order to proceed with the reduction of the evolution equations (1.7), which corresponds to the situation at hand, we will need some additional assumptions relative to the constitutive function λ ; specifically, the hypotheses on λ which will hold throughout the rest of this section are

$$(A1) \quad \lambda \in C^1(R^3; [0, \infty)), \lambda(\zeta) > 0, \forall \zeta \neq 0$$

$$(A2) \quad |\lambda^*(\zeta)| < \infty, \forall \zeta \in R^1$$

$$(A3) \quad 0 < \zeta \lambda^*(\zeta) + \dot{\lambda}(\zeta) < \infty, \forall \zeta \in R^1$$

where $\dot{\lambda}(\zeta) \equiv \lambda((0, \zeta, 0))$, $\zeta \in R^1$. By (A1) and the definition of $\dot{\lambda}$ it is immediate that $\dot{\lambda} \in C^1(R^1; [0, \infty))$.

We now proceed with the reduction of the nonlinear evolution equations (1.7). In view of (1.5₁), (1.19), in Ω

$$(0, E_2, 0) = \lambda(\underline{D})(D_1, D_2, D_3)$$

from which it follows that, in Ω , $D_1 = D_3 = 0$ and $E_2(x_1, t) = \lambda(\underline{D})D_2(x_1, x_2, x_3, t)$.

However, $\operatorname{div} \underline{D} = \frac{\partial D_2}{\partial x_2} = 0$ so that, for each $t \geq 0$, D_2 can depend, at most, on x_1, x_3 . As E_2 depends only on x_1

$$\begin{aligned} \frac{\partial E_2}{\partial x_3} &= \frac{\partial}{\partial x_3} (\lambda(\underline{D})D_2(x_1, x_3, t)) \\ &= \frac{\partial}{\partial x_3} (\lambda(0, D_2, 0)D_2(x_1, x_3, t)) \\ &= \frac{\partial}{\partial x_3} (\dot{\lambda}(D_2(x_1, x_3, t))D_2(x_1, x_3, t)) \\ &= \frac{\partial D_2}{\partial x_3} (\dot{\lambda}'(D_2)D_2 + \dot{\lambda}(D_2)) = 0. \end{aligned}$$

By hypothesis (A3) it then follows that $\frac{\partial D_2}{\partial x_3} = 0$ and, thus, in Ω ⁽¹⁾

$$(1.20) \quad E(x, t) = (0, D_2(x_1, t), 0)$$

In view of (1.20), not only is $\operatorname{div} \underline{D} = 0$ automatically satisfied in Ω , but,

(1) At this point hypothesis (A3) could be weakened to the assumption that $(\dot{\lambda}^*(\zeta))' \neq 0$ a.e. on R^1 and (1.20) would still obtain.

as is easily verified, so are the nonlinear evolution equations (1.7) for $i = 1, 3$, i.e.,

$$\begin{aligned} \frac{\partial}{\partial x_i} (\text{grad } \lambda(\underline{D}) \cdot \underline{D}) &= \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_2} \dot{\lambda}(D_2(x_1, t)) \cdot D_2(x_1, t) \right) \\ &= 0, \quad i = 1, 2, 3 \end{aligned}$$

while $(D_i)_{tt} = v^2(\lambda(\underline{D})D_i) \equiv 0$ for $i = 1, 3$. For $i = 2$ we then obtain, for $-\infty < x_1 < \infty$, and $0 \leq t \leq T$,

$$\begin{aligned} (1.21) \quad \mu \frac{\partial^2 D_2}{\partial t^2}(x_1, t) &= v^2[\dot{\lambda}(D_2(x_1, t)) D_2(x_1, t)] \\ &= \frac{\partial^2}{\partial x_1^2} [\dot{\lambda}(D_2(x_1, t)) D_2(x_1, t)]. \end{aligned}$$

In view of our assumption that the rod is infinite in extent, the boundary conditions (1.16) do not come into play here. In fact to simplify the analysis we will now assume that the initial data $D_2(x_1, 0)$ and $\frac{\partial D_2}{\partial t}(x_1, 0)$, $-\infty < x_1 < \infty$, have compact support on \mathbb{R}^1 . Then, in view of the fact that hypothesis (A3) implies the strict hyperbolicity of (1.21), $D_2(x_1, t)$ will also have compact support on \mathbb{R}^1 , say $\text{supp } D_2 \subset (-\delta, \delta)$, for some δ , $0 < \delta < \infty$, for as long as a smooth solution of the initial-value problem for (1.21) exists.

We now set $x_1 \equiv x$, $D_2 \equiv u$. Then, for the physical situation described above, the initial-boundary value problem associated with the coupled system of nonlinear evolution equations (1.7) reduces to the following nonlinear, one-dimensional, initial value problem on the x axis: find $u = u(x, t)$, $-\infty < x < \infty$, $0 \leq t \leq T$, such that

$$(1.22) \quad \begin{cases} \mu \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} (u \dot{\lambda}(u)), & (x, t) \in (-\infty, \infty) \times [0, T] \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), & -\infty < x < \infty \end{cases}$$

where $\mu > 0$, λ^* satisfies the hypotheses (A1) - (A3), and $u_0(x), v_0(x)$ have compact support on \mathbb{R}^1 . Any smooth solution of (1.22) will also have compact support on \mathbb{R}^1 and, as indicated above, we will take the support of u to be in the interval $(-\delta, \delta)$ where $0 < \delta < \infty$.

2. Global Nonexistence of Electric Induction Fields

In this section we will demonstrate that under the additional hypothesis on the constitutive function $\lambda^*(\zeta) = \lambda((0, \zeta, 0))$,⁽²⁾

(λ4) For all $\zeta \in \mathbb{R}^1$ and some $\alpha > 2$

$$\alpha \int_0^\zeta \rho \lambda^*(\rho) d\rho \geq \zeta^2 \lambda^*(\zeta),$$

smooth global solutions of (1.22), i.e., solutions of (1.22) on $(-\infty, \infty) \times [0, T)$, for all $T > 0$, will not, in general, exist; in fact, we will show that under relatively mild assumptions on the initial data, the $L^2(-\infty, \delta)$ norm of $u(x, t)$ must be bounded from below by a real-valued nonnegative function of t which becomes infinite as $t \rightarrow t_\infty < \infty$. Some growth estimates for solutions of the initial-boundary value problem (1.24), which are valid on the maximal time-interval of existence, will also be derived. In §3, we show that under stronger assumptions on $\lambda^*(\zeta)$, than that represented by (λ4), it is possible to demonstrate via a Riemann Invariant argument that smooth solutions of (1.22) cannot exist globally due to finite-time breakdown of the space time gradient $(u_x(x, t), u_t(x, t))$.

Before proceeding with the analysis, let us note that if we set $\psi(\zeta) = \zeta \lambda^*(\zeta)$, $\zeta \in \mathbb{R}^1$, and $\Sigma(\zeta) = \int_0^\zeta \psi(\rho) d\rho$ then $\zeta \lambda^*(\zeta) = \Sigma'(\zeta)$ and hypothesis (λ4) is equivalent to

(λ4) For all $\zeta \in \mathbb{R}^1$ and some $\alpha > 2$ $\alpha \Sigma(\zeta) \geq \zeta \Sigma'(\zeta)$.

The proof of the global nonexistence results referred to above now proceeds via a series of lemmas, the first of which is just an energy conservation theorem for the solutions of (1.24). Thus, let δ , $0 < \delta < \infty$, be any constant such that

⁽²⁾ While this hypothesis is satisfied by $\lambda^*(\zeta) = \text{const.}$ none of our results apply to the linear wave equation, i.e., see the footnote following Theorem III.

$\bar{\delta} > \delta$ where, by assumption, $\text{supp } u(x,t) \subset (-\delta, \delta)$ then we have

Lemma 2. If we define the total energy $E(t)$ of the system (1.22) by

$$(2.1) \quad E(t) = \frac{\mu}{2} \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx + \int_{-\infty}^{\bar{\delta}} \left(\int_0^{u(x,t)} \rho \lambda^*(\rho) d\rho \right) dx,$$

then for as long as smooth solutions of (1.24) exist,

$$(2.2) \quad E(t) = \frac{\mu}{2} \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x v_0(y) dy \right)^2 dx + \int_{-\infty}^{\bar{\delta}} \left(\int_0^{u_0(x)} \rho \lambda^*(\rho) d\rho \right) dx$$

Proof. In view of the definitions of $\psi(\zeta)$, $\Sigma(\zeta)$,

$$(2.3) \quad E(t) = \frac{\mu}{2} \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx + \int_{-\infty}^{\bar{\delta}} \Sigma(u(x,t)) dx$$

Therefore,

$$\begin{aligned} (2.4) \quad \dot{E}(t) &= \mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right) \left(\int_{-\infty}^x u_{tt}(y,t) dy \right) dx \\ &\quad + \int_{-\infty}^{\bar{\delta}} \Sigma'(u(x,t)) u_t(x,t) dx \\ &= \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right) \left(\int_{-\infty}^x \frac{\partial^2}{\partial y^2} \psi(u(y,t)) dy \right) dx \\ &\quad + \int_{-\infty}^{\bar{\delta}} \Sigma'(u(x,t)) u_t(x,t) dx \\ &= \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right) \psi(u(x,t))_{,x} dx \\ &\quad + \int_{-\infty}^{\bar{\delta}} \Sigma'(u(x,t)) u_t(x,t) dx \end{aligned}$$

where we have used (1.22) and the compact support of $u(x,t)$ on \mathbb{R}^1 , i.e.,

$$\begin{aligned} \int_{-\infty}^x \frac{\partial^2}{\partial y^2} \psi(u(y,t)) dy &= \psi(u(y,t))_{,y} \Big|_{-\infty}^x \\ &= \psi(u(x,t))_{,x} \\ &\quad - \lim_{\rho \rightarrow -\infty} (\psi(u(y,t))_{,y} \Big|_{\rho}) \\ &\quad - \lim_{\rho \rightarrow 0} (\psi(u(y,t))_{,y} \Big|_{\rho}) \\ &\equiv \psi(u(x,t))_{,x} \end{aligned}$$

as $\psi(0) = 0$ by virtue of (λ_1) and the definition of ψ . Therefore:

$$(2.5) \quad \dot{E}(t) = \int_{-\infty}^{\bar{\delta}} \frac{\partial}{\partial x} [\psi(u(x,t)) \int_{-\infty}^x u_t(y,t) dy] dx \\ - \int_{-\infty}^{\bar{\delta}} \psi(u(x,t)) u_{xx}(x,t) dx + \int_{-\infty}^{\bar{\delta}} \Sigma'(u(x,t)) u_t(x,t) dx = 0$$

as $\psi(\zeta) = \Sigma'(\zeta)$, $\forall \zeta \in \mathbb{R}^1$, by definition, $\psi(0) = 0$, and $\text{supp } u \subset (-\delta, \delta)$, $\delta < \bar{\delta}$. Equation (2.2) then follows by integration over $[0, t]$, the definition of $E(t)$, and the initial conditions. Q.E.D.

Our next lemma is concerned with establishing a certain differential inequality for a real-valued nonnegative functional defined on solutions $u(x, t)$ of the initial-boundary value problem (1.22); namely, we have

Lemma 3. Let $u(x, t)$, $(x, t) \in (-\infty, \infty) \times [0, T]$ be a smooth solution of (1.24) and define

$$(2.6) \quad F(t) = \mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u(y, t) dy \right)^2 dx + \beta (t + t_0)^2$$

where $\beta, t_0 \geq 0$. If $\lambda(\zeta)$ satisfies $(\lambda_1) - (\lambda_4)$, then for $0 \leq t < T$

$$(2.7) \quad F'' \geq (\gamma + 1) F'^2 \geq -2(\gamma + 1) F (\beta + 2E(0))$$

where $\gamma = \frac{\alpha - 2}{4} > 0$ (with α the constant which arises in the constitutive assumption (λ_4)) and $E(0)$, the initial energy, is given by the right-hand side of (2.2).

Proof. By direct differentiation we have

$$(2.8) \quad F'(t) = 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u(y,t) dy \right) \left(\int_{-\infty}^x u_t(y,t) dy \right) dx + 2\beta(t+t_0)$$

and

$$(2.9) \quad F''(t) = 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx \\ + 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u(y,t) dy \right) \left(\int_{-\infty}^x u_{tt}(y,t) dy \right) dx + 2\beta.$$

Again, in view of (1.24₁), the definition of $\psi(\zeta)$, $\zeta \in \mathbb{R}^1$, and the compact support of u , we have

$$(2.10) \quad F''(t) = 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx \\ + 2 \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u(y,t) dy \right) \psi(u(x,t))_{,x} dx + 2\beta \\ = 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx \\ + 2 \int_{-\infty}^{\bar{\delta}} \frac{\partial}{\partial x} \left\{ \left(\int_{-\infty}^x u(y,t) dy \right) \psi(u(x,t)) \right\} dx \\ - 2 \int_{-\infty}^{\bar{\delta}} u(x,t) \psi(u(x,t)) dx + 2\beta \\ = 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx \\ - 2 \int_{-\infty}^{\bar{\delta}} u(x,t) \Sigma'(u(x,t)) dx + 2\beta$$

By adding and subtracting $2\alpha \int_{-\infty}^{\bar{\delta}} \Sigma(u(x,t)) dx$ on the right-hand side of the last line in (2.10) we obtain

$$(2.11) \quad F''(t) = 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx - 2\alpha \int_{-\infty}^{\bar{\delta}} \Sigma(u(x,t)) dx \\ + 2 \int_{-\infty}^{\bar{\delta}} \left(\alpha \Sigma(u(x,t)) - u(x,t) \Sigma'(u(x,t)) \right) dx + 2\beta \\ \geq 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx - 2\alpha \int_{-\infty}^{\bar{\delta}} \Sigma(u(x,t)) dx$$

where we have used the hypothesis (A4) in the form given by (A4). However, in

view of the definitions of $E(t)$, i.e. (2.1), and $\Sigma(t)$, $t \in \mathbb{R}^+$, the inequality in (2.11) may be replaced by

$$\begin{aligned}
 (2.12) \quad F'''(t) &\geq 2\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx \\
 &\quad - 2\alpha \left[E(t) - \frac{\mu}{2} \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx \right] + 2\beta \\
 &= (2+\alpha)\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx \\
 &\quad - 2\alpha E(0) + 2\beta
 \end{aligned}$$

where we have used the energy conservation result of Lemma 2. Finally, we rewrite the last inequality in (2.12) in the form

$$\begin{aligned}
 (2.13) \quad F'''(t) &\geq (2+\alpha) \left[\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx + \beta \right] \\
 &\quad - \alpha [\beta + 2E(0)]
 \end{aligned}$$

Combining (2.8), (2.13) and (2.6) we now obtain

$$\begin{aligned}
 (2.14) \quad FF'' &= \left(\frac{\alpha+2}{4} \right) F'^2 \\
 &\geq (2+\alpha) \left[\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u(y,t) dy \right)^2 dx + \beta(t+t_0)^2 \right] \\
 &\quad \times \left[\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx + \beta \right] \\
 &\quad - \alpha \Gamma(\beta + 2E(0)) \\
 &= (2+\alpha) \left[\mu \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u(y,t) dy \right) \left(\int_{-\infty}^x u_t(y,t) dy \right) dx \right. \\
 &\quad \left. + \beta(t+t_0)^2 \right]^2
 \end{aligned}$$

$$\begin{aligned}
&= (2+\alpha) \left\{ \left(\mu \int_{-\infty}^{\delta} \left(\int_{-\infty}^x u(y,t) dy \right)^2 dx + \beta(t + t_0) \right) \right. \\
&\quad \times \left. \left(\mu \int_{-\infty}^{\delta} \left(\int_{-\infty}^x u_t(y,t) dy \right)^2 dx + \beta \right) \right. \\
&\quad - \left. \left(\mu \int_{-\infty}^{\delta} \left(\int_{-\infty}^x u(y,t) dy \right) \left(\int_{-\infty}^x u_t(y,t) dy \right) dx \right. \right. \\
&\quad \left. \left. + \beta(t + t_0) \right)^2 \right\} \\
&= \alpha F(\beta + 2E(0)).
\end{aligned}$$

By virtue of the Cauchy-Schwarz inequality the $\{ \}$ expression in the last inequality in (2.14) is nonnegative for all t , $0 \leq t < T$, and, therefore,

$$(2.15) \quad FF'' - \left(\frac{\alpha+2}{\mu} \right) F'^2 \geq -\alpha F(\beta + 2E(0)), \quad 0 \leq t < T$$

The required result, i.e., (2.7) now follows directly from (2.15) if we set $\gamma = (\alpha-2)/4$.

Q.E.D.

Global nonexistence of solutions to the initial-boundary value problem (1.22) can now easily be shown to be a consequence of the differential inequality (2.7) under various assumptions on the initial energy $E(0)$ and the initial data $u_0(x)$, $v_0(x)$. To simplify the discussion we introduce the notation

$$(2.16a) \quad I(u_0) = \mu \int_{-\infty}^{\delta} \left(\int_{-\infty}^x u_0(y) dy \right)^2 dx$$

$$(2.16b) \quad J(u_0, v_0) = 2\mu \int_{-\infty}^{\delta} \left(\int_{-\infty}^x u_0(y) dy \right) \left(\int_{-\infty}^x v_0(y) dy \right) dx$$

Our first series of results (Theorems I and II) concern situations in which $e(0) \leq 0$; while these results may be of some interest for their own sake they are not relevant to the example in which, paralleling the assumptions in Townes, et. al., [14], Strauss [15], and Whitham [16] we have

$$\lambda^*(\epsilon) = \lambda_0^* + \lambda^*, \epsilon^{1/2}; \quad \lambda_0^* > 0, \quad \lambda^* > 0$$

(we must have $E(0) > 0$ for any such λ^*); results concerning the development of singularities in the gradient (u_x, u_t) for this situation, in which γ is not genuinely nonlinear, are presented in §3 as an application of some recent work of Klainerman and Majda [21].

Theorem 1. Let $u(x,t)$ be a solution of (1.4) with $\gamma > 0$ and let the function $\lambda^*(\zeta) = \lambda((0,\zeta,0))$ satisfies $(\lambda 1) - (\lambda 4)$. If $\int_{-\infty}^{\infty} u_0(x) dx > 0$ and

$$(2.17) \quad \int_{-\infty}^{\infty} \left(\int_0^{\bar{\delta}} u_0(x) \psi(\rho) d\rho \right) dx \leq - \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^x v_0(y) dy \right)^2 dx$$

then $\kappa = \kappa(u; \delta) > 0$ and $t_{\infty} < \infty$ such that

$$(2.18) \quad \|u(t)\|_{L^2(-\infty, \delta)}^2 \geq \kappa G(t), \quad 0 \leq t \leq t_{\max}$$

where $[0, t_{\max})$ denotes the maximal interval of existence of $u(x,t)$ and

$$\lim_{t \rightarrow t_{\infty}} G(t) = +\infty.$$

Remark: As a consequence of (2.18), and the fact that $G(t)$ tends to $+\infty$ as $t \rightarrow t_{\infty}$, it follows that $t_{\max} \leq t_{\infty}$, i.e., that the maximal interval of existence of $u(x,t)$ is finite; it may happen, however, that $t_{\max} < t_{\infty}$ and thus, without stronger assumptions on $\lambda^*(\zeta)$, we cannot conclude that $u(x,t)$, or its derivatives, blow-up in an appropriate norm as $t \rightarrow t_{\infty}$. (See e.g. Ball [10], [11] for a relevant discussion of the relationship between global nonexistence and finite-time blow-up theorems for solutions of nonlinear evolution equations).

Proof (Theorem 1). In view of (2.17), $E(0) \leq 0$. Thus, if we set $\beta = 0$ in (2.7) this inequality reduces to

$$(2.19) \quad F_0(t) \Gamma_0''(t) - (\gamma+1) \Gamma_0'^2(t) \geq 0, \quad 0 \leq t \leq t_{\max},$$

$F_0(t) = \mu \int_{-\infty}^{\infty} \left(\int_{-\infty}^x u(y,t) dy \right)^2 dx$. But (2.19) is easily seen to be equivalent to

$$(2.20) \quad (\Gamma_0^{-\gamma})''(t) \leq 0, \quad 0 \leq t \leq t_{\max}$$

Two successive integrations of (2.20) yield

$$(2.21) \quad F_0^{-Y}(t) \leq -Y F_0^{-Y-1}(0) F_0'(0)t + F_0^{-Y}(0), \quad 0 \leq t \leq t_{\max}$$

or, as $Y > 0$, $F_0(t) > 0$

$$(2.22) \quad F_0(t) \geq \left[\frac{F_0^Y(0)}{1 - Y \left(\frac{F_0'(0)}{F_0^Y(0)} \right) t} \right]^{\frac{1}{Y}} \equiv G(t), \quad 0 \leq t \leq t_{\max}$$

Clearly, $\lim_{t \rightarrow t_{\infty}} G(t) = +\infty$ where

$$t_{\infty} = \frac{1}{Y} \left(\frac{F_0(0)}{F_0'(0)} \right) = \frac{1}{Y} \frac{F(u_0)}{J(u_0, v_0)} < \infty$$

Also, as $\text{supp } u \subset (-\delta, \delta)$ and $\bar{\delta} > \delta$

$$\begin{aligned} (2.23) \quad & \int_{-\infty}^{\bar{\delta}} \left(\int_{-\infty}^x u(y, t) dy \right)^2 dx = \int_{-\infty}^{\bar{\delta}} \left(\int_{-\bar{\delta}}^x u(y, t) dy \right)^2 dx \\ & \leq \int_{-\bar{\delta}}^{\bar{\delta}} (x + \bar{\delta}) \left(\int_{-\bar{\delta}}^x u^2(y, t) dy \right) dx \\ & \leq \left(\int_{-\bar{\delta}}^{\bar{\delta}} (x + \bar{\delta})^2 dx \right)^{1/2} \left(\int_{-\bar{\delta}}^{\bar{\delta}} \left(\int_{-\bar{\delta}}^x u^2(y, t) dy \right)^2 dx \right)^{1/2} \\ & = (2\bar{\delta})^{3/2} \left(\int_{-\bar{\delta}}^{\bar{\delta}} \left(\int_{-\bar{\delta}}^{\bar{\delta}} u^2(y, t) dy \right)^2 dx \right)^{1/2} \\ & = (2\bar{\delta})^{3/2} \cdot \sqrt{2\bar{\delta}} \int_{-\bar{\delta}}^{\bar{\delta}} u^2(y, t) dy = 4\bar{\delta}^2 \int_{-\bar{\delta}}^{\bar{\delta}} u^2(y, t) dy \end{aligned}$$

and therefore,

$$(2.24) \quad \|u(t)\|_{L^2(-\bar{\delta}, \bar{\delta})}^2 \geq \frac{1}{4\bar{\delta}^2} \int_0^L \left(\int_{-\infty}^x u(y, t) dy \right)^2 dx.$$

The growth estimate (2.18), valid for $0 \leq t \leq t_{\max}$, with $k = \frac{1}{4u\delta^2}$, now follows directly from (2.22), (2.24), the definition of $F_0(t)$, the fact that

$u(x,t) \equiv 0$, $-\infty < x < -\bar{\delta}$, and the observation that (2.17), (2.23) and, thus, (2.18) also hold as we let $\bar{\delta} \rightarrow \delta$. Therefore, if $\text{supp } u \subset (-\delta, \delta)$ then for $0 \leq t < t_{\max}$ $\|u(t)\|_{L^2(-\infty, \delta)}^2 \geq k(u; \delta)G(t)$ with $G(t)$ given by (2.22). Q.E.D.

There are several other situations in which the same basic conclusion, as that expressed by Theorem I, follows; we will examine two such sets of circumstances below which correspond to situations in which we have, respectively, $J(u_0, v_0) = 0$ and $J(u_0, v_0) < 0$, with $E(0) < 0$ in both cases. Suppose, first of all, that $E(0) < 0$ with $v_0(x) \equiv 0$, $-\infty < x < \delta$; in this case we may choose $\beta = \beta_0$ such that $2E(0) + \beta_0 = 0$ and therefore (2.7) reduces to (2.19) with $F_0(t)$ replaced by

$$F(t; \beta_0, t_0) \equiv u \int_0^L \left(\int_{-\infty}^x u(y, t) dy \right)^2 dx + \beta_0 (t + t_0)^2$$

Therefore, $F(t; \beta_0, t_0)$ satisfies, for $0 \leq t \leq t_{\max}$,

$$(2.25) \quad F(t; \beta_0, t_0) \geq \left[\frac{F^\gamma(0; \beta_0, t_0)}{1 - \gamma \left(\frac{F'(0; \beta_0, t_0)}{F(0; \beta_0, t_0)} \right) t} \right]^{\frac{1}{\gamma}} \equiv H(t),$$

so that $\lim_{t \rightarrow t_\infty(t_0)} H(t) = +\infty$ where

$$(2.26) \quad t_\infty(t_0) = \frac{1}{\gamma} \frac{F(0; \beta_0, t_0)}{F'(0; \beta_0, t_0)} \\ = \frac{1}{\gamma} \left(\frac{I(u_0) + \beta_0 t_0^2}{2\beta_0 t_0} \right)$$

We note that, in view of our hypothesis,

$$(2.27) \quad \beta_0 = -2 \int_{-\infty}^{\bar{\delta}} \left(\int_0^x u_0(\rho) \psi(\rho) d\rho \right) dx > 0, \quad \bar{\delta} > \delta$$

It is not difficult to show that the minimum value of $t_\infty(t_0)$ is achieved at

$$t_0 = \tilde{t}_0 \equiv \sqrt{\frac{I(u_0)}{\beta}}$$

and that

$$(2.28) \quad t_{\infty}(\tilde{t}_0) = \sqrt{\frac{I(u_0)}{\beta_0}} = \tilde{t}_0$$

Choosing $t_0 = \tilde{t}_0$ in (2.25) we have, therefore,

$$(2.29) \quad \begin{aligned} & u \int_0^L \left(\int_{-\infty}^x u(y, t) dy \right)^2 dx + \beta_0 (t + \tilde{t}_0)^2 \\ & \geq \left[\frac{(I(u_0) + \beta_0 \tilde{t}_0^2)^Y}{1 - \frac{1}{t_{\infty}(\tilde{t}_0)}} \right]^{\frac{1}{Y}} \\ & \geq \frac{I(u_0)}{\left(1 - \sqrt{\frac{\beta_0}{I(u_0)}} t \right)^{\frac{1}{Y}}} \end{aligned}$$

for $0 \leq t \leq t_{\max}$. In view of (2.24), (2.29) we then have the growth estimate (let $\bar{\delta} + \delta$)

$$(2.30) \quad \begin{aligned} & \frac{1}{K} \|u(t)\|_{L^2(-\infty, \delta)}^2 + \beta_0 \left(t + \sqrt{\frac{I(u_0)}{\beta_0}} \right)^2 \\ & \geq \frac{I(u_0)}{\left(1 - \sqrt{\frac{\beta_0}{I(u_0)}} t \right)^{\frac{1}{Y}}}, \end{aligned}$$

for $0 \leq t \leq t_{\max} \leq \sqrt{\frac{I(u_0)}{\beta_0}}$, where β_0 is given by (2.27) with δ replaced by $\bar{\delta}$ and $I(u_0)$ by (2.16a), with δ in place of $\bar{\delta}$. The estimate (2.30) establishes global nonexistence of solutions to the initial-boundary value problem, under the hypotheses $(\lambda 1) - (\lambda 4)$, for the case where the initial data satisfy

$$v_0(x) \geq 0, \quad -\infty < x < \delta \quad \text{and} \quad \int_{-\infty}^{\delta} \left(\int_0^{u_0(x)} \rho^*(s) ds \right) dx < 0.$$

Having examined the cases where $E(0) \leq 0$ with $I(u_0, v_0) > 0$ and $E(0) < 0$ with $J(u_0, v_0) = 0$, $v_0(x) \equiv 0$, $-\infty < x < \delta$, we now want to look at the situation where $E(0) < 0$, i.e.,

$$(2.31) \quad \int_{-\infty}^{\delta} \left(\int_0^{u_0(x)} \psi(\rho) d\rho \right) dx < - \frac{u}{2} \int_{-\infty}^{\delta} \left(\int_{-\infty}^x v_0(y) dy \right)^2 dx$$

and $J(u_0, v_0) < 0$. In this case we may again choose $\beta = \bar{\beta}_0$ such that $2E(0) + \bar{\beta}_0 = 0$, so that $F(t; \bar{\beta}_0, t_0)$ satisfies (2.25), with $\beta_0 = \bar{\beta}_0$, for $0 \leq t \leq t_{\max}$. We note that we now have

$$(2.32) \quad t_{\infty}(t_0) = \frac{I(u_0) + \bar{\beta}_0 t_0^2}{2\bar{\beta}_0 t_0 - |J(u_0, v_0)|}$$

where

$$(2.33) \quad \bar{\beta}_0 = -u \int_{-\infty}^{\delta} \left(\int_{-\infty}^x v_0(y) dy \right)^2 dx - 2 \int_{-\infty}^{\delta} \int_0^{u_0(x)} \rho \lambda(\rho) d\rho dx > 0$$

and thus we must choose $t_0 \geq \bar{t}_0$ where

$$(2.34) \quad \bar{t}_0 > \frac{1}{2\bar{\beta}_0} |J(u_0, v_0)|$$

It is a relatively simple matter to show that $t_{\infty}(t_0)$ achieves a minimum at

$$(2.35) \quad t_0 = \bar{t}_0 = \frac{1}{2\bar{\beta}_0} \left(|J(u_0, v_0)| + \sqrt{J^2(u_0, v_0) + 4\bar{\beta}_0 I(u_0)} \right)$$

If we denote $t_{\infty}(\bar{t}_0) = \bar{t}_{\infty}$ then we have the estimate

$$(2.36) \quad \begin{aligned} & u \int_{-\infty}^{\delta} \left(\int_{-\infty}^x u(y, t) dy \right)^2 dx + \bar{\beta}_0 (+ + \bar{t}_0)^2 \\ & \geq \left[\frac{(I(u_0) + \bar{\beta}_0 \bar{t}_0^2)^{\gamma}}{1 - \bar{t}_{\infty}^{-1} t} \right]^{\frac{1}{\gamma}} \geq \frac{I(u_0)}{(1 - \bar{t}_{\infty}^{-1} t)^{1/\gamma}} \end{aligned}$$

for $0 \leq t \leq t_{\max}$ and the companion estimate

$$(2.37) \quad \frac{1}{K} \|u(t)\|_{L^2(-\infty, \delta)}^2 + \beta_0(t + \bar{t}_0)^2 \geq \frac{I(u_0)}{(1 - \bar{t}_\infty^{-1} t)} \frac{1}{Y}$$

for $0 \leq t \leq t_{\max} \leq \bar{t}_\infty$ and global nonexistence of solutions to the initial-boundary value problem (1.22) follows as in the previous cases. We may summarize the two results corresponding to the situation where $E(0) < 0$ as

Theorem II. Let $u(x, t)$ be a solution of (1.22) and assume that the constitutive function $\lambda^*(\rho) = \lambda((0, \rho, 0))$ satisfies $(\lambda 1) - (\lambda 4)$. Then

$$(i) \quad \text{If } v_0(x) \equiv 0, \quad -\infty < x < \delta \quad \text{and} \quad \int_{-\infty}^{\delta} \left(\int_0^{u_0(x)} \rho \lambda^*(\rho) d\rho \right) dx < 0,$$

then $u(x, t)$ satisfies, for $0 \leq t \leq t_{\max} \leq \sqrt{\frac{I(u_0)}{\beta_0}}$, the growth estimate (2.30) where β_0 is given by (2.27) with δ in place of $\bar{\delta}$.

$$(ii) \quad \text{If the initial data } (u_0(x), v_0(x)) \text{ satisfy (2.31) and}$$

$$\int_{-\infty}^{\delta} \left(\int_{-\infty}^x u_0(y) dy \right) \left(\int_{-\infty}^x v_0(y) dy \right) dx < 0,$$

then $u(x, t)$ satisfies, for $0 \leq t \leq t_{\max} \leq \bar{t}_\infty$, the growth estimate (2.37), where \bar{t}_0 is given by (2.23), \bar{t}_c by (2.35), (2.16b), and $\bar{t}_\infty = t_\infty(\bar{t}_0)$ where $t_\infty(t_0)$ is given by (2.32). In both cases (i) and (ii) above the respective estimates (2.30), (2.37) imply that solutions of (1.24) cannot exist globally, i.e., for $t \in [0, \infty)$.

We now want to consider situations in which

$$e(0) > 0 \quad (\text{i.e., } \lambda^*(\zeta) = \lambda_0^* + \lambda_2^* \zeta^2, \quad \lambda_0^* > 0, \quad \lambda_2^* > 0).$$

Theorem III. Let $\lambda^*(\zeta)$ satisfy $(\lambda 1) - (\lambda 3)$ and define $\mathcal{J}(u_0)$, $\mathcal{J}(u_0, v_0)$ as in Theorem I. If $e(0) > 0$ with $\mathcal{J}(u_0, v_0) > 0$ and

$$\frac{\mathcal{J}^2(u_0, v_0)}{\mathcal{J}(u_0)} > 8e(0) \quad (3)$$

then no smooth solution of (1.24) can exist for all $t \in [0, \infty)$.

Proof. Assume that a smooth solution does exist on $[0, \infty)$. Then (2.7)

holds $\forall t$ and we rewrite it as (set $\beta = 0$)

$$(2.38) \quad FF'' - (\gamma + 1)F'^2 \geq -2v^2(2\gamma + 1)F$$

where $v^2 = 2e(0) > 0$.

As $F'(0) = \mathcal{J}(u_0, v_0) > 0$ we have

$$(F^{-\gamma})'(0) = -\gamma F^{-(\gamma+1)}(0)F'(0) < 0.$$

By continuity $(F^{-\gamma})'(t) < 0$ for t sufficiently small. If $(F^{-\gamma})'(t) \not< 0$ for as long as smooth solutions exist then $\exists t = t^*$ such that

$$F^{-\gamma}(t) < 0, \quad t < t^* \text{ but } (F^{-\gamma})'(t^*) = 0.$$

We will show that this can not happen. Since $F(t) > 0$, $t \in [0, t^*]$ may rewrite (2.38) as

$$(2.39) \quad (F^{-\gamma})'' \leq 2\gamma v^2(2\gamma + 1)F^{-(\gamma+1)}; \quad t \in [0, t^*]$$

On $[0, t^*)$, $(F^{-\gamma})'(t) < 0$. Multiply (2.39) thru by $(F^{-\gamma})'(t)$, $t \in [0, t^*]$ to obtain

$$2(F^{-\gamma})''(F^{-\gamma})' \geq 4\gamma v^2(2\gamma + 1)F^{-(\gamma+1)}(F^{-\gamma})'$$

or

(3) This condition is easily seen to require that $\int_0^t \left(\int_0^{u_0(x)} \rho \lambda^*(\rho) d\rho \right) dx$ be sufficiently negative; thus neither this result, or any of our other results of a similar nature, apply to the linear wave equation obtained by taking $\lambda^*(t) = \text{const.}$

$$\frac{d}{dt} [(F^{-Y})']^2 \geq 4Y^2 v^2 \{ -(2Y+1) F^{-2(Y+1)} F' \} = 4Y^2 v^2 \frac{d}{dt} F^{-(2Y+1)}$$

We now integrate this last estimate over $[0, t]$, $t \in [0, t^*]$ so as to obtain

$$(2.40) \quad [(F^{-Y})'(t)]^2 - 4Y^2 v^2 F^{-(2Y+1)}(t) \geq \kappa_0$$

where

$$(2.41) \quad \kappa_0 = Y^2 F^{-(2Y+1)}(0) [F^{-1}(0) F'^2(0) - 4v^2] > 0$$

by our hypothesis on the initial data ($v^2 = 2\ell(0)$). We now, factor the left-hand side of (2.40) and write it as

$$(2.42) \quad \{ (F^{-Y})'(t) - 2Yv[F^{-(2Y+1)}(t)]^{1/2} \} \times \\ \{ (F^{-Y})'(t) + 2Yv[F^{-(2Y+1)}(t)]^{1/2} \} \geq \kappa_0 > 0.$$

Since $(F^{-Y})'(t) \leq 0$, $t \in [0, t^*]$ the first factor in (2.40) is negative for $t \in [0, t^*]$ and thus $\forall t \in [0, t^*]$ the second factor is also negative. Thus

$$(2.43) \quad (F^{-Y})'(t^*) < -2Yv[F^{-(2Y+1)}(t^*)]^{1/2}$$

Hence $\exists t^*$ such that $(F^{-Y})'(t^*) = 0$ and thus $(F^{-Y})'(t) < 0$ for as long as smooth solutions exist, which implies the estimate

$$(2.44) \quad [(F^{-Y})'(t)]^2 \geq \kappa_0 + 4Y^2 v^2 F^{-(2Y+1)}(t)$$

This last estimate is valid $\forall t$ - as long as smooth solutions exist. Therefore

$$(2.45) \quad [-Y F^{-(Y+1)} F']^2 \geq \kappa_0 + 4Y^2 v^2 F^{-(2Y+1)}(t)$$

As $-Y F^{-(Y+1)} F' < 0$, taking the square root on both sides of (2.45) yields

$$(2.46) \quad |-Y F^{-(Y+1)} F'| \geq (\kappa_0 + 4Y^2 v^2 F^{-(2Y+1)})^{1/2}$$

or

$$(2.47) \quad -\gamma F^{-(\gamma+1)} F' \leq -(\kappa_0 + 4\gamma^2 v^2 F^{-(\gamma+1)})^{1/2}$$

But this clearly implies that

$$(2.48) \quad F'(t) \geq (4\gamma^2 F(t) + \kappa_0 \gamma^{-2} F^{2(\gamma+1)}(t))^{1/2}$$

for as long as smooth solutions exist. Hence

$$(2.49) \quad \int \frac{F(t)}{F(0)(4v^2 G + \kappa_0 \gamma^{-2} G^{2(\gamma+1)})^{1/2}} dG \geq t$$

which implies a finite time of existence for any smooth solution since the integral on the left hand side of (2.49) is convergent $\forall F$.

Our last result is a growth estimate for smooth solutions of (1.24) which is valid on $[0, t_{\max})$, the estimate shows that under certain conditions on the data, $\|u\|_{L^2(0,L)}^2$ must grow quadratically in time.

Theorem III. Let $u(x,t)$ be a solution of (1.24) with $\lambda^*(\zeta)$ satisfying $(\lambda_1) - (\lambda_2)$. Then if $\varepsilon(0) > 0$, $\rho(u_0, v_0) > 0$ with $\int_{-\infty}^{\infty} u_0(x) v_0(x) dx > \sqrt{2\varepsilon(0)} (\int_{-\infty}^{\infty} u_0^2(x) dx)^{1/2}$ we must have on $0 \leq t < t_{\max}$

$$(2.50) \quad (4v_0^2) \|u\|_{L^2(0,L)}^2 \geq \mathcal{A}(u_0) + 2^{3/2} \sqrt{\varepsilon(0)} \sqrt{\mathcal{A}(u_0)} t + 2\varepsilon(0) t^2$$

Proof. We begin with (2.35), i.e.

$$FF'' - (\gamma+1)F^{1/2} \geq 2v^2(2\gamma+1)F, \quad 0 \leq t < t_{\max}$$

where $v^2 = 2\varepsilon(0) > 0$.

By our hypotheses: $F(0) > 0$, hence $\exists \eta > 0$ s.t.

$$F(t) > 0, \quad t \in [0, \eta]$$

We multiply the differential inequality (2.38) through by

$$-v(F^{-v})'(t)(F^{-(v+2)}(t))'' , t \in [0, \eta]$$

and integrate over $[0, t]$, $t < \eta$ so as to obtain

$$(2.51) \quad [(F^{-v})'(t)]^2 - 4v^2 v^2 F^{-(2v+1)}(t) \geq [(F^{-v})'(0)]^2 - 4v^2 v^2 F^{-(2v+1)}(0) > 0$$

by virtue of the definition of $F(t)$ and the hypothesis relative to the initial data. Factoring both sides of (14) we have

$$(2.52) \quad [(F^{-v})'(t) - 2v v F^{-(v+1/2)}(t)][(F^{-v})'(t) + 2v v F^{-(v+1/2)}(t)] \\ \geq [(F^{-v})'(0) - 2v v F^{-(v+1)}(0)][(F^{-v})'(0) + 2v v F^{-(v+1/2)}(0)]$$

and thus as

$$(F^{-v})'(t) = -v F^{(v+1)} F'(t) < 0 , t \in [0, \eta] ,$$

$$(2.53) \quad (F^{-v})'(t) < -2v v F^{-(v+1/2)}(t) , t \in [0, \eta]$$

Hence, by continuity we can not have $\dot{F}(\eta) = 0$, for any $\eta > 0$. Thus $\dot{F}(t) > 0$, $0 \leq t < t_{\max}$ and (2.53) holds for all t , $0 \leq t \leq t_{\max}$.

From (2.52) we obtain directly the estimate

$$(2.54) \quad F(t) \geq (vt + F^{1/2}(0))^2 , 0 \leq t < t_{\max}$$

and the quadratic growth estimate now follows from the definition of $F(t)$

and the estimate

$$\|u\|_{L^2(-\infty, \delta)}^2 \geq \frac{1}{4\mu\delta^2} \int_0^\delta \left(\int_{-\infty}^x u(y, t) dy \right)^2 dx$$

3. Riemann Invariants and Finite-Time Breakdown of the Electric Induction Field.

In this section we offer a brief demonstration of the fact that under a slightly different set of assumptions on $\lambda^*(\zeta), \zeta \in \mathbb{R}^1$, other than those represented by (A1) - (A4), it is possible in certain situations to apply the Riemann invariant argument of Lax [13] so as to conclude that finite-time breakdown of $u_t + \frac{1}{\sqrt{\mu}} \sqrt{\psi'(u)} u_x$ must occur, where $u(x,t)$ is a solution of the initial-boundary value problem (1.22); the requisite assumptions on $\lambda^*(\zeta)$, however, cannot be realized in the case where $\lambda^*(\zeta) = \lambda_0^* + \lambda_2^* \zeta^2$, $\lambda_0^* > 0$, $\lambda_2^* > 0$. In this latter situation it is possible, however, to apply some recent results of Klainerman and Majda [21] as we will indicate below.

In [13] Lax considers the nonlinear initial-boundary value problem on

$$(3.1) \quad \begin{cases} y_{tt}(x,t) = K^2(y_x) y_{xx}(x,t), \\ y(x,0) = y_0(x), y_t(x,0) = 0; 0 \leq x \leq L \\ y(0,t) = y(L,t) = 0, t > 0 \end{cases}$$

This problem may be extended to a pure-initial value problem on $\mathbb{R}^1 \times [0, \infty)$ by extending $y_0(\cdot), y(\cdot, t)$ as odd functions to $(-L, L)$ and then periodically, to all of \mathbb{R}^1 , with period $2L$. By setting $U = y_x, V = y_t$ the resulting extended initial-value problem on \mathbb{R}^1 is then easily seen to be equivalent to a pure initial-value problem for a coupled quasilinear system on $\mathbb{R}^1 \times [0, \infty)$, i.e.,

$$(3.2) \quad \begin{pmatrix} U \\ V \end{pmatrix}_{,t} + \begin{pmatrix} 0 & -1 \\ -K^2(U) & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_{,x} = 0$$

$$\begin{pmatrix} U(x,0) \\ V(x,0) \end{pmatrix} = \begin{pmatrix} \tilde{y}_0'(x) \\ 0 \end{pmatrix} \quad \left\{ \begin{array}{l} \tilde{y}_0 \text{ the extension} \\ \text{of } y_0 \text{ to } \mathbb{R}^1 \end{array} \right\}$$

The eigenvalues and eigenvectors associated with the system (3.2) are, respectively,

$$(3.3) \quad \begin{Bmatrix} \delta(U) \\ \rho(U) \end{Bmatrix} = \pm K(U) \quad \text{and} \quad \begin{bmatrix} 1 \\ \pm K(U) \end{bmatrix}$$

and thus the system is hyperbolic if and only if $K^2(\xi) > 0, \forall \xi \in \mathbb{R}^1$. Also, the system may be diagonalized in a familiar way so as to yield the system

$$(3.4) \quad \begin{aligned} V' + K(U)U' &= 0 \\ V'' - K(U)U'' &= 0 \end{aligned}$$

where $' \equiv \frac{\partial}{\partial t} - K(U) \frac{\partial}{\partial x}$ and $'' \equiv \frac{\partial}{\partial t} + K(U) \frac{\partial}{\partial x}$ denote, respectively, differentiation along the right and left-hand characteristics defined by the ordinary differential equations $\frac{dx}{dt} = \pm K(U)$. Using (3.4) one then shows in the standard way that the Riemann Invariants

$$(3.5) \quad \begin{aligned} R(U,V) &= V + \int_0^U K(\zeta) d\zeta \\ S(U,V) &= V - \int_0^U K(\zeta) d\zeta \end{aligned}$$

satisfy $R' = S'' = 0$, i.e., that they are constant along the respective characteristic curves. It is shown in [13] that with a suitable choice of $H = H(R,A)$, the function $Z = \exp(A)R_x$ satisfies $Z' = -[(\exp(-H))\delta_R]Z^2$ where $\delta_R = \frac{\partial K}{\partial U} / K(U)$ so that Z , and hence R_x , must breakdown (blow-up) in finite time if $\exists C > 0$ such that $|(\exp(-H)\delta_R)| \geq C$; this last condition, on the other hand, turns out to be a consequence of the assumption that $\exists \bar{\epsilon} > 0$ such that $|\partial K / \partial U| \geq \bar{\epsilon} > 0, \forall U \in \mathbb{R}^1$. Finite-time breakdown for R_x then implies finite-time breakdown for at least one of the second-order derivatives v_{xx}, v_{tt} of the solution $y(x,t)$ to the nonlinear initial-boundary value problem (3.1), as

$$\begin{aligned} R_x &= R_U U_x + R_V V_x \\ &= K(U)U_x + V_x \\ &= K(y_x)y_{xx} + y_{xt} \end{aligned}$$

Suppose that we now reconsider the initial value problem (1.22) and recall that as a consequence of the fact that $\text{supp } u(x,t) \subset (-\zeta, \zeta)$

$$\mu \int_{-\infty}^x u_{tt}(y,t) dy = \psi(u(x,t))_{,x} = \psi'(u) u_x(x,t), \quad \text{for } -\infty < x < \infty \text{ where}$$

$$\psi(\zeta) = \zeta \lambda^*(\zeta), \quad \forall \zeta \in \mathbb{R}^1. \quad \text{If we set}$$

$$(3.6) \quad v(x,t) = \int_{-\infty}^x u_t(y,t) dy, \quad t \geq 0$$

then, clearly, $v_x(x,t) = u_t(x,t)$ and $v_t(x,t) = \int_{-\infty}^x u_{tt}(y,t) dy = \frac{1}{\mu} \psi'(u) u_x(x,t)$.

Also, $u(x,0) = u_0(x)$, $v(x,0) = \int_{-\infty}^x u_t(y,0) dy = \int_{-\infty}^x v_0(y) dy$. Therefore, the initial-boundary value problem (1.22) for $u(x,t)$ is easily seen to be equivalent to the following initial-boundary value problem for the pair $(u(x,t), v(x,t))$:

$$(3.7) \quad \begin{cases} u_t - v_x = 0 \\ v_t - \frac{1}{\mu} \psi'(u) u_x = 0 \end{cases} \quad \begin{cases} -\infty < x < \infty \\ t \geq 0 \end{cases}$$

$$u(x,0) = u_0(x), \quad v(x,0) = \int_{-\infty}^x v_0(y) dy, \quad -\infty < x < \infty$$

$$u(0,t) = u(L,t) = 0, \quad t \geq 0$$

The system (3.7) is clearly of the same form as that considered by Lax [13], i.e., (3.2), if we assume that $v_{ij}(x) \equiv 0$, $-\infty < x < \infty$.

Thus in standard matrix form our system (3.7) is

$$(3.8) \quad \begin{cases} \begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -\frac{1}{u}\psi'(u) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0 & \begin{cases} -\infty < x < \infty \\ t \geq 0 \end{cases} \\ \begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} \tilde{u}_0(x) \\ 0 \end{pmatrix}, & -\infty < x < \infty \end{cases}$$

In comparing (3.8) with (3.2) we clearly have the correspondence

$K^2(\zeta) = \frac{1}{u} \psi'(\zeta)$, $\zeta \in \mathbb{R}^1$, and thus (3.8) is a hyperbolic system if and only if

$$\psi'(\zeta) = \zeta \tilde{\lambda}'(\zeta) + \tilde{\lambda}(\zeta) > 0, \quad \forall \zeta \in \mathbb{R}^1$$

which is precisely hypothesis (λ3). The Riemann Invariants associated with the system (3.8) are, clearly, given by the expressions

$$(3.9) \quad \begin{cases} r(u,v) = v + \frac{1}{\sqrt{u}} \int_0^u \sqrt{\psi'(\rho)} d\rho \\ s(u,v) = v - \frac{1}{\sqrt{u}} \int_0^u \sqrt{\psi'(\rho)} d\rho \end{cases}$$

and they satisfy $r' = s' = 0$ along the respective characteristics given by

$$\frac{dx}{dt} = \pm \sqrt{\frac{\psi'(u)}{u}} \quad \text{where} \quad r' \equiv \frac{\partial}{\partial t} - \sqrt{\frac{\psi'(u)}{u}} \frac{\partial}{\partial x} \quad \text{and} \quad s' \equiv \frac{\partial}{\partial t} + \sqrt{\frac{\psi'(u)}{u}} \frac{\partial}{\partial x}.$$

By the results in [13], which we have described above, finite-time breakdown (blow-up) of

$$(3.10) \quad \begin{aligned} r_x &= v_x + \frac{1}{\sqrt{u}} \sqrt{\psi'(u)} u_x \\ &= u_t + \frac{1}{\sqrt{u}} \sqrt{\psi'(u)} u_x \end{aligned}$$

will occur if, $\forall \zeta \in \mathbb{R}^1$ (this is actually needed only $\forall \zeta$ sufficiently small, if the initial data is small, as we indicate below),

$$\begin{aligned} \left| \frac{d}{d\zeta} \left(\frac{1}{\sqrt{\mu}} \cdot \sqrt{\psi'(\zeta)} \right) \right| &= \frac{1}{2} \left| \frac{-\psi''(\zeta)}{\sqrt{\mu\psi'(\zeta)}} \right| \\ &= \frac{1}{2\sqrt{\mu\psi'(\zeta)}} \cdot |\psi''(\zeta)| \geq \epsilon^* \end{aligned}$$

for some $\epsilon^* > 0$. Using the relationship between $\ddot{\lambda}(\zeta)$ and $\psi(\zeta)$ this last condition is equivalent to the requirement that $\ddot{\lambda}(\zeta), \forall \zeta \in \mathbb{R}^1$, satisfy, for some $\epsilon > 0$,

$$(15) \quad |\zeta \ddot{\lambda}''(\zeta) + 2\ddot{\lambda}'(\zeta)| \geq \epsilon \sqrt{\zeta \ddot{\lambda}'(\zeta) + \ddot{\lambda}(\zeta)},$$

It also follows from the work of Lax [13] that

$$t_{\max} \approx \frac{4\sqrt{\mu}}{\max u_0(x)} \cdot \frac{\sqrt{\psi'(0)}}{\psi''(0)}$$

As $\psi'(\zeta) = \zeta \ddot{\lambda}'(\zeta) + \ddot{\lambda}(\zeta)$ and $\psi''(\zeta) = \zeta \ddot{\lambda}''(\zeta) + 2\ddot{\lambda}'(\zeta), \forall \zeta \in \mathbb{R}^1$, we clearly must require that $\ddot{\lambda}(\zeta)$ satisfy

$$(16) \quad 0 < \ddot{\lambda}(0) < \infty, \quad 0 < \ddot{\lambda}'(0) < \infty, \quad |\ddot{\lambda}''(0)| < \infty,$$

in which case $t_{\max} = \frac{2\sqrt{\mu}}{\max u_0(x)} \cdot \left(\frac{\ddot{\lambda}(0)}{\ddot{\lambda}'(0)} \right)$. Now, it is a simple matter to show that the a priori estimate $\|u(x,t)\| + \|\delta(x,t)\| \leq \sup_x |u(x,0)| + \sup_x |\delta(x,0)|$ holds on $(-\infty, \infty) \times [0, t_{\max}]$. By virtue of the definitions (3.9) similar a priori estimates hold for $u(x,t), v(x,t)$,

and thus $|u(x,t)|$ and $|v(x,t)|$ will be small on $(-\epsilon, \epsilon) \times (0, t_{\max})$ if $\sup_x |u(x,0)|$ is sufficiently small. It then follows that (25) need only hold for some $\epsilon > 0$ and all $\zeta \in \mathbb{R}^1$ which are sufficiently small in magnitude. Now, if $\lambda^*(\zeta) = \lambda_0^* + \lambda_1^* \zeta$, $\lambda_0^* > 0$, $\lambda_1^* > 0$, $\epsilon > 0$, then (25) becomes

$$(3.11) \quad 2\lambda_1^* |\zeta| \geq \epsilon \sqrt{2\lambda_1^* \zeta + \lambda_0^*}$$

which is certainly satisfied for $|\zeta| \sim A$, $A > 0$, if $\epsilon > 0$ is chosen sufficiently small. In this case $\psi(\zeta) = \lambda_0^* \zeta + \lambda_1^* \zeta^2$ so that $\psi''(0) = 2\lambda_1^* \neq 0$ and the system (3.8) is genuinely nonlinear. However, if $\lambda^*(\zeta) = \lambda_0^* + \lambda_2^* \zeta^2$, $\lambda_0^* > 0$, $\lambda_2^* > 0$ then (25) becomes

$$(3.12) \quad 6\lambda_2^* |\zeta| \geq \epsilon \sqrt{3\lambda_2^* \zeta^2 + \lambda_0^*}$$

which cannot be satisfied for any $\epsilon > 0$ even as $|\zeta| \rightarrow 0$. The essential problem here is the lack of genuine nonlinearity vis a vis $\psi(\zeta)$ which in this case is $\psi(\zeta) = \lambda_0^* \zeta + \lambda_2^* \zeta^3$ (so that $\psi'(\zeta) = 6\lambda_2^* \zeta^2$ which vanishes at $\zeta = 0$). In order to obtain finite-time breakdown of smooth solutions to initial-value problems for the quasilinear system (3.7) on \mathbb{R}^1 , when the data $u(x,0), v(x,0)$ has compact support in \mathbb{R}^1 , and $\psi(\zeta) = \zeta \lambda^*(\zeta) = \lambda_0^* \zeta + \lambda_2^* \zeta^3$, $\lambda_0^* > 0$, $\lambda_2^* > 0$, we may appeal to a recent result of S. Klainerman and A. Majda [21]. Stated in terms of the initial-value problem for the diagonal system associated with the Riemann invariants, i.e.,

$$(3.13) \quad \begin{cases} r' = \frac{\partial r}{\partial t} + \sqrt{\frac{\psi'}{\mu}} \frac{\partial r}{\partial x} = 0 \\ s' = \frac{\partial s}{\partial t} + \sqrt{\frac{\psi'}{\mu}} \frac{\partial s}{\partial x} = 0 \end{cases}$$

where κ, δ are given by (3.9), the relevant result in [21] says that any C^1 solution of (3.13) with C^1 initial data $\kappa_0(x) = \kappa(x, 0)$, $\delta_0(x) = \delta(x, 0)$, having compact support in \mathbb{R}^1 , must develop singularities in the first derivatives κ_x, δ_x in finite time provided $\psi'(\zeta)$ is not constant on any open interval; this last condition is, of course, equivalent to having $\psi''(\zeta) \neq 0$ on every open interval - which is certainly true for the cubic $\psi(\zeta)$ of our example. It is worth noting that as with the work of Lax [13], the work of Klainerman and Majda [21] was done with the nonlinear wave equation in (3.1) in mind and, as in [13], the development of singularities in the first derivatives κ_x, δ_x leads to the prediction that solutions of the initial-value problem for (3.1) with C^2 initial data having compact support in \mathbb{R}^1 must develop singularities in the second derivatives y_{xx}, y_{xt} . However, as is the case with the work in [13], the results in [21] now predict that solutions of the initial-value problem (1.22), with C^1 initial-data having compact support in \mathbb{R}^1 , must develop singularities in the first derivatives u_x, u_t .

Remarks. In closing we offer a few comments concerning the problem of proving global nonexistence of smooth electric induction fields of the form (1.20) in a finite rod occupying the configuration $0 \leq x_1 \leq L$. The relevant one-dimensional equation is still (1.21) but now we must take account of the implications of the boundary conditions (1.16), assuming as before, that the rod is embedded in a perfect conductor. At the planar boundary at $x_1 = 0$, $\underline{v} = (-1, 0, 0)$, $\underline{t} = (0, 1, 0)$ and thus by (1.16a) with $x_1 = 0$, $f(x_2, x_3) = C_1$

$$(3.14) \quad \underline{D}(\underline{x}, \underline{t}) \cdot \underline{v} \Big|_{\substack{x_1=0 \\ t>0}} = [(0, D_2(x_1, t), 0) \cdot (-1, 0, 0)] \Big|_{\substack{x_1=0 \\ t>0}} = 0$$

so that $\sigma(x,t)$, the surface charge density at $x_1 = 0$ must vanish for all $t > 0$. An analogous result holds at $x_1 = L$ where $v = (1,0,0)$. In order to satisfy the boundary condition (1.16b) along the planar face at $x_1 = 0$, for $t > 0$, we require that

$$\begin{aligned}
 (3.15) \quad & \lambda(D(x,t))D(x,t) \cdot v \Big|_{\substack{x_1=0 \\ t>0}} \\
 &= (0, \lambda^*(D_2(x_1,t))D_2(x_1,t), 0) \cdot (0,1,0) \Big|_{\substack{x_1=0 \\ t>0}} \\
 &= \lambda^*(D_2(x_1,t))D_2(x_1,t) \Big|_{\substack{x_1=0 \\ t>0}} \\
 &= 0,
 \end{aligned}$$

from which it follows that $D_2(0,t) = 0$, $t > 0$. In an analogous manner we have $D_2(L,t) = 0$. In place of the initial-value problem (1.22) for $u(x,t) = D_2(x_1,t)$ we then have the initial-boundary value problem

$$(3.15) \quad \begin{cases} u \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} (u \lambda^*(u)), & (x,t) \in [0,L] \times [0,T), \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), & 0 \leq x \leq L \end{cases}$$

and, in addition, because of the embedding of the rod in a perfect conductor, $\text{supp } u \subseteq [0,L]$. The principal difficulty that arises in trying to apply the analysis of both this and the previous section to either (3.16) or the pure initial value problem that results by making the usual extensions of the initial data, first to $[-L,L]$ and then to all of \mathbb{R}^1 with period $2L$, revolves around dealing with the integral $\int_{-\infty}^x u_{\tau t}(v,t) dv$ which, in the analysis of the

infinite rod, is equal to $\frac{1}{\mu} \psi'(u) u_x(x,t)$. If, as is customary in trying to prove breakdown of smooth solutions, we assume that $u(\cdot, t)$ is of class C^2 , for all $t > 0$, on \mathbb{R}^1 then integration across the planar boundary at $x_1 = 0$ forces upon the analysis the a priori assumption that not only $u(0, t) = 0$, $t > 0$, but also $u_x(0, t) = u_{xx}(0, t) = 0$, $t > 0$; it is unlikely that any classical solution of (3.16) could exist under such circumstances.⁽⁴⁾ We hope to address the problem of global nonexistence of smooth solutions to the initial-boundary value problem (3.16) in a future paper.

⁽⁴⁾ The author is indebted to Prof. Morton Gurtin for this observation.

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